Pythagorean Tuning

Pythagorean tuning is not only mathematically elegant, but also easy to tune by ear, especially in a time before electronic tuners. It builds all intervals using simple whole-number ratios—primarily based on the numbers 2 and 3 (specifically the ratio 3:2 for the perfect fifth). It was commonly used during the medieval and Renaissance periods, from roughly the 13th to the 16th centuries

It is a form of **just intonation** that emphasizes relationships derived from powers of 3 and 2. The system yields uses intervals such as:

- Perfect fifths (3:2)
- Perfect fourths (4:3)
- Major seconds (9:8)
- Minor sevenths (16:9)

These are all rational-number ratios, which means they relate closely to natural harmonics and blend well acoustically. Medieval theorists explained Pythagorean tuning using the ratios 12:9:8:6, which represent a series of simple intervals—like fifths and fourths.

Unlike equal temperament, which divides the octave into 12 equal steps, Pythagorean tuning does not evenly split the octave. Instead, the notes are constructed from stacked perfect fifths (multiplying or dividing by 3/2). Because of this, the scale contains slight discrepancies between the same notes.

For instance:

- D♯ is not the same as E♭ in Pythagorean tuning when it is based on A.
- D♯ is derived by stacking fifths downward, while E♭ is found by stacking upward, and due to the math of powers of 3/2, these two notes are close but not identical. This difference is known as the Pythagorean comma (~23.5 cents).

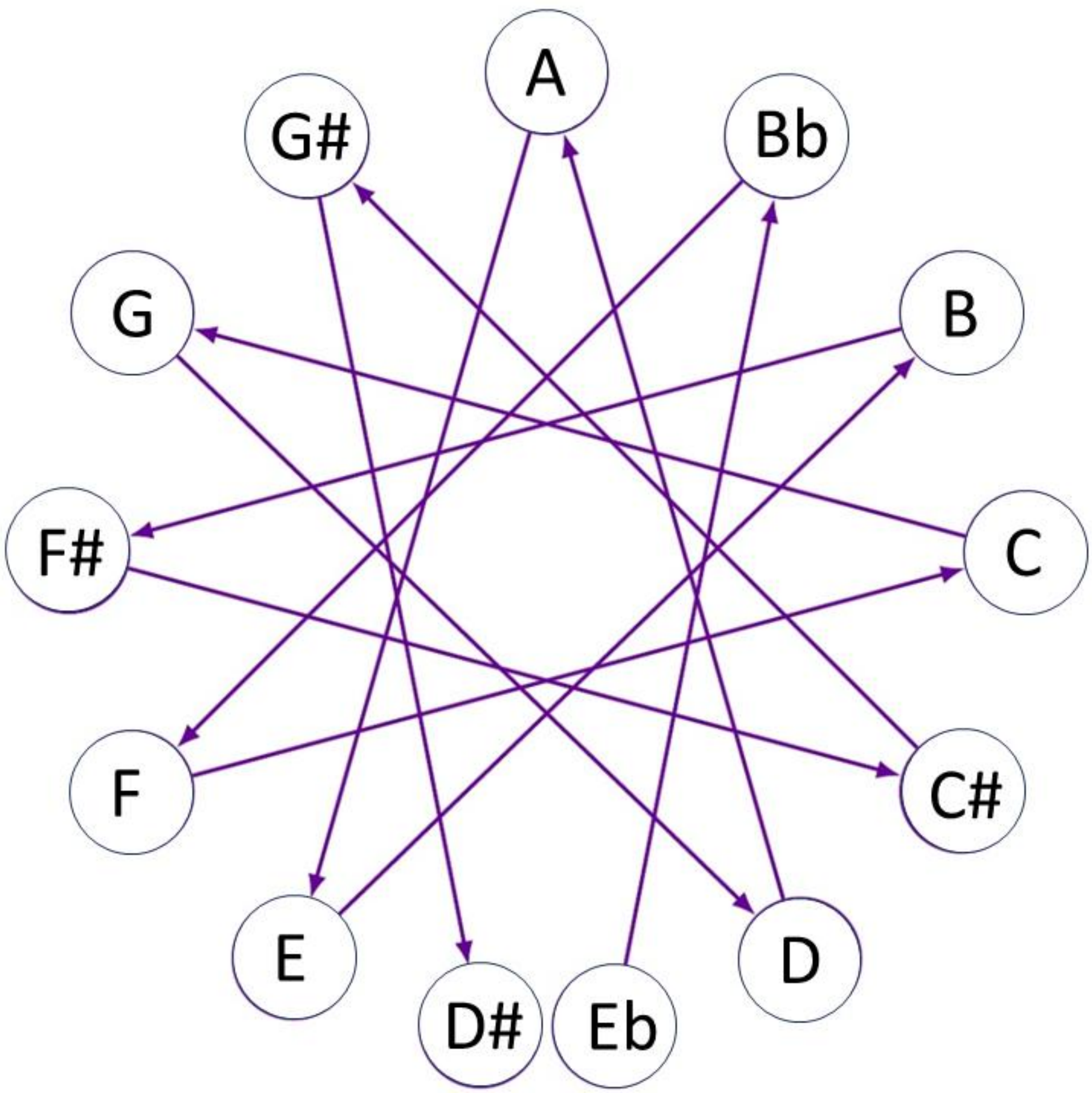
The one potential flaw of this system is that the fourth or fifth between the extreme notes of the series, G#-Eb, will be out of tune: in the colorful language of intonation, a "wolf" interval.

In Pythagorean tuning, the augmented fourth (A4) is slightly higher in pitch than the diminished fifth (d5). While both intervals are considered a tritone in equal temperament (where they're enharmonically the same), Pythagorean tuning distinguishes between them. This small but audible difference illustrates how tuning affects how in tune the music sounds when it is transposed or modulates through different keys.

To build a complete 12-note chromatic scale—the kind found on keyboards by around 1300—musicians stacked 11 pure fifths in a sequence like this:

E♭ → B♭ → F → C → G → D → A → E → B → F♯ → C♯ → G♯ (→ D♯)

This series uses the 3:2 ratio repeatedly to generate each new note. The D♯ is an enharmonic extension—it completes the spiral of fifths and nearly matches the starting E♭, though not exactly (because of the **Pythagorean comma**). In reality the final notes, D# and Eb, are not as far apart as in the diagram. If accurately shown would be nearly overlapping.



Pythagorean Chromatic Scale Based on A440

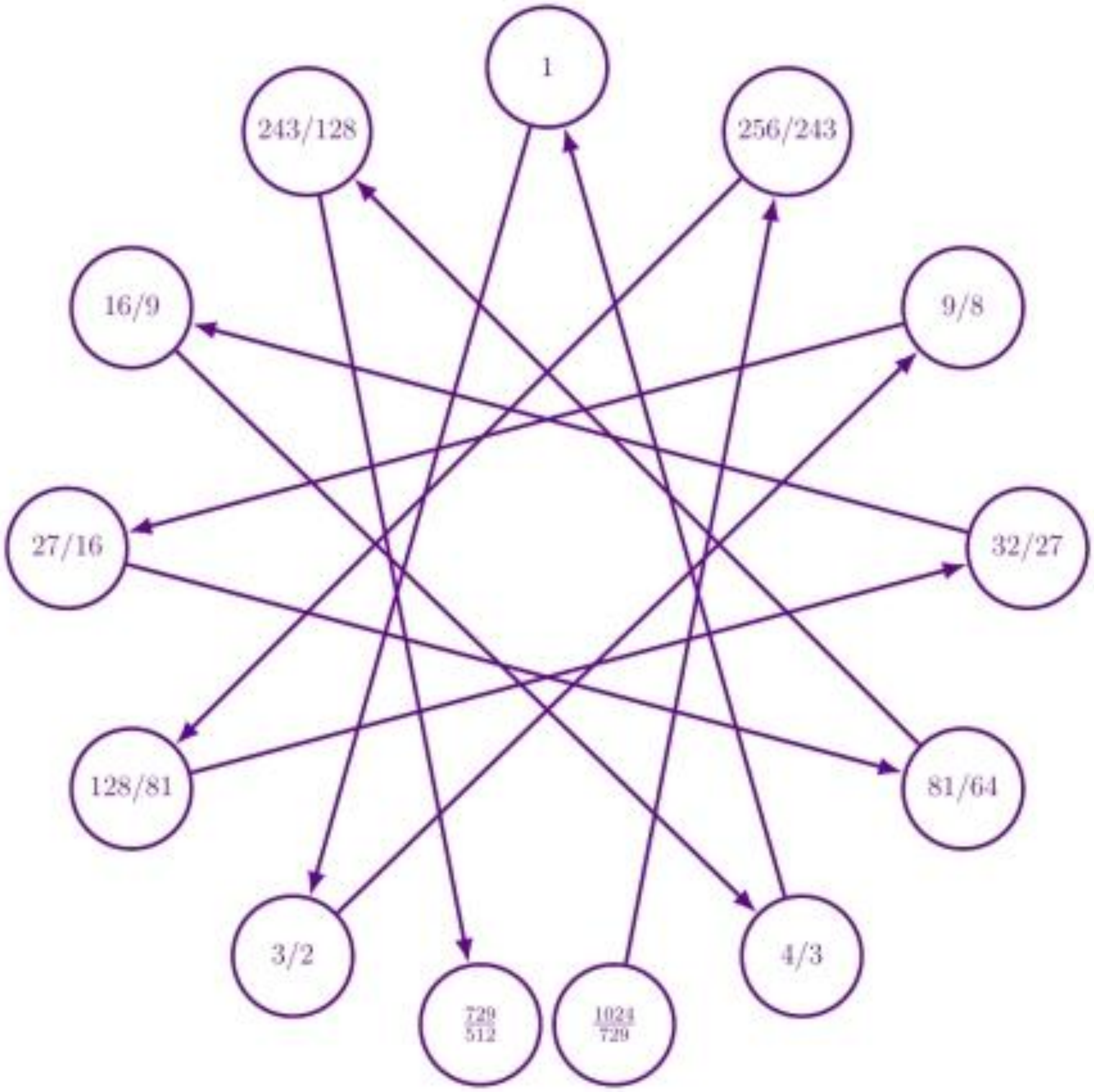
Note Name	Ratio	HZ
Eb	1:1	313.24
E	256:243	330.00
F	9:8	347.65
F# / Gb	32:27	371.25
G	81:64	391.11
G# / Ab	4:3	417.66
A	729:512	440.00
A# / Bb	3:2	463.54
B	128:81	495.00
C	27:16	521.48
C# / Db	16:9	556.88
D	243:128	586.67
D#	2:1	618.05

The 13 Intervals of Medieval Music

According to **Anonymous I (c. 1290)**, medieval theorists recognized 13 distinct intervals between the unison and the octave. These include:

Interval	**Ratio**	**Cents**
Unison	1:1	0.00
Minor Second	256:243	90.22
Major Second	9:8	203.91
Minor Third	32:27	294.13
Major Third	81:64	407.82
Fourth	4:3	498.04
Augmented Fourth	729:512	611.73
Fifth	3:2	701.96
Minor Sixth	128:81	792.18
Major Sixth	27:16	905.87
Minor Seventh	16:9	996.09
Major Seventh	243:128	1109.78
Octave	2:1	1200.00

The ratios or multipliers used to raise a note by a given musical interval in the Pythagorean scale. Here it is shown on our familiar diagram:


1
256/243
9/8
32/27
81/64
4/3
1024/729
729/512
3/2
128/81
27/16
16/9
243/128

Pythagorean Scale Construction (Starting from C)

We'll build a C major scale using Pythagorean tuning, where all notes are derived from stacking perfect fifths (ratio 3:2) and perfect fourths (ratio 4:3).

Let's begin with C as the reference pitch (frequency = 261 Hz for example), and construct the rest of the notes in the scale by moving up and down in fifths and fourths.

Step-by-Step Construction

We'll begin with C (261 Hz) and generate the rest by stacking fifths (×3/2) and descending fourths (÷4/3 or ×3/4).

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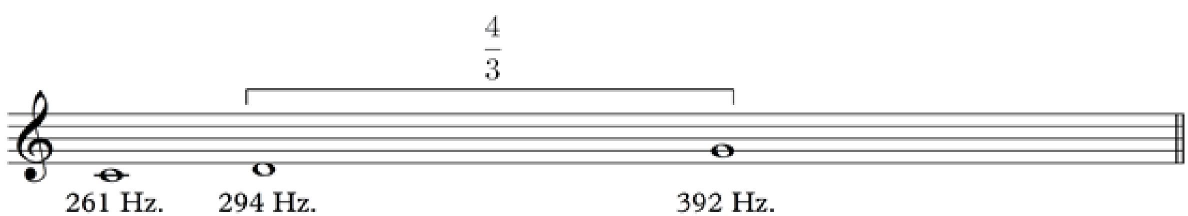
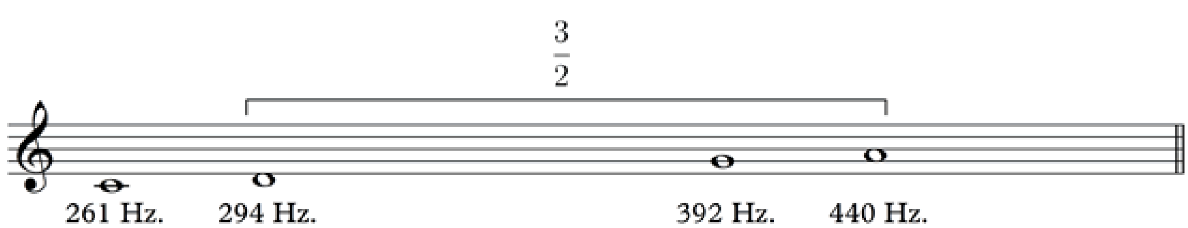
1. C = 261 Hz (Base)
2. G (a fifth above C)

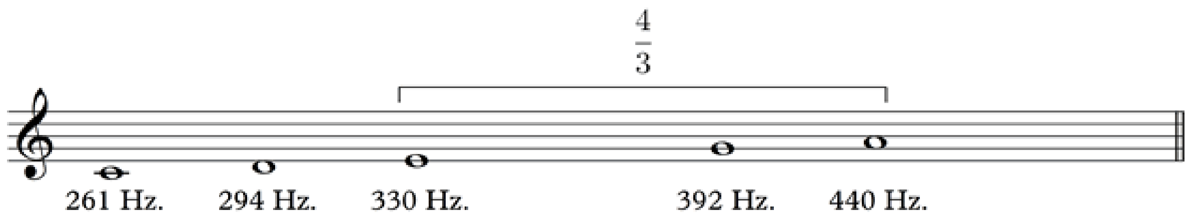
3. D (a fourth below G)



4. A (a fifth above D)



5. E (a fourth below A)



6. B (a fifth above E)

→ E × 3 / 2 = 330.328 × 3 / 2 = 495.49 Hz

7. F♯ (a fifth above B)

→ B × 3 / 2 = 495.49 × 3 / 2 = 743.235 Hz

Reverse Process for Remaining Notes

To generate the flats and remaining diatonic notes, you can also go down by fifths (or up by fourths):

- F: a fifth below C → 261 × 2 / 3 = 174 Hz
- B♭: a fifth below F → 174 × 2 / 3 = 116 Hz
- E♭: a fifth below B♭
- A♭: a fifth below E♭
- D♭: a fifth below A♭
- G♭: a fifth below D♭

At this point, you'd need to adjust the octaves by transposing some down by octaves (÷2, ÷4, etc.) to fit them within a single octave

This is our familiar diagram when the frequency we arbitrarily called 1 is the note called C: